Confinement and stability of the motion of particles and photons in branes: a geometrical approach

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Let us praise geometry

(for the role it has played so far in the formulation of modern theories of Physics)
“The unreasonable effectiveness of Mathematics in the Natural Sciences”

Eugene Wigner
Dialogue between C. N. Yang and S. S. Chern:
Yang: “On se sent ému et stupéfait de ce que vous, les mathematiciens, ayez inventé ces notions à partir de rien”

Chern: “Non, non! Ces notions ne sont pas du tout inventées. Elles existent toutes réeellement.”

(G. Lochak, La géométrisation de la physique, Flammarion, 1994)
Geometrization of Physics: General Relativity

Inexhaustible source of new ideas for Geometry

Albert Einstein (1879-1905)
For example:
b) The theory of connections (Gerhard Hessenberg, 1916)

b) *Der Ricci-Kalkül* (Schouten, 1924), the first book on connections.

c) It is worth of noting that all the first papers on connection theory mention the General Theory of Relativity: Weyl, Levi-Civita, Schouten, Struik, Élie Cartan, and others.
The so-called *post-Riemannian geometry* very often had as its main motivation the invention/discovery of new structures capable of geometrize the electromagnetic field.

H. Gönner, *On the history of unified field theories*  

Two typical examples of post-Riemannian geometry:

i) Einstein-Cartan theory (1924)
   - introducing the concept of torsion;


ii) Weyl geometry (1918) – in which the Levi-Civita connection is generalized.
The Kaluza-Klein Miracle

- The idea of higher dimensions: Kaluza-Klein theory
- (1921-1926) introduces a new way of geometrizing the electromagnetic field.
Higher-dimensional spacetime theories:

- The seminal character of Kaluza-Klein theory
Superstring theory (10 or 26 dimensions)

M-theory (which unifies five superstring theories)

Induced matter theory (IMT) (5 dimensions)

Randall-Sundrum (RS) (5 dimensions)
In IMT and RS scenarios spacetime is usually represented as a hypersurface \( S \) (the brane) locally and isometrically embedded in a five-dimensional space (bulk).

- Ricci-flat (IMT) or AdS (RS)
In the Randall-Sundrum model massive particles and photons are confined to the hypersurface $S$.

The mechanism that is responsible for the confinement is of a quantum nature: one postulates the existence of a scalar field which interacts with the particles.

An important feature of the Randall-Sundrum model is the assumption that the bulk has the geometry of a warped product space (WPS).
What is a warped product space?

(Bishop and O’Neill (1969))
Definition: Let \((M,h)\) and \((N,k)\) be two Riemannian (or pseudo-Riemannian) manifolds of dimension \(m\) and \(n\), respectively. Suppose we are given a differentiable function from \(N\) to \(R\) (a “warping function”). Then we construct a new Riemannian (pseudo-Riemannian) manifold by setting \(M \times N\) and defining a metric

\[
g = e^{2f} h \otimes k
\]

In local coordinates we have

\[
dS^2 = g_{ab} dy^a dy^b.
\]
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\[ \frac{d^2 y^a}{d\lambda^2} + (5) \Gamma^a_{bc} \frac{dy^b}{d\lambda} \frac{dy^c}{d\lambda} = 0, \quad (1) \]

\[ \frac{d^2 x^\mu}{d\lambda^2} + (4) \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = \phi^\mu, \quad (1) \]

\[ \phi^\mu = -(5) \Gamma^\mu_{44} \left( \frac{dl}{d\lambda} \right)^2 - 2(5) \Gamma^\mu_{\alpha4} \frac{dx^\alpha}{d\lambda} \frac{dl}{d\lambda} \]

\[ - \frac{1}{2} g^\mu_{\alpha4} \left( g_{4\alpha,\beta} + g_{4\beta,\alpha} - g_{\alpha\beta,4} \right) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}, \quad (1) \]

\[ dS^2 = e^{2f} h_{\alpha\beta} dx^\alpha dx^\beta - dl^2, \quad (1) \]
\[ \frac{d^2 l}{d\lambda^2} + f' e^{2f} h_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0. \] (1)

\[ \frac{d^2 l}{d\lambda^2} + f' \left( 1 + \left( \frac{dl}{d\lambda} \right)^2 \right) = 0. \] (1)

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Confinement and stability of the motion: All information is contained in the warping function.

Centre: Particles enter and leave the hypersurface. We have a sort of “oscillatory” confinement.
Phase diagram: picturing the oscillatory confinement (quasi-confinement) in a neighbourhood of S.
Instable equilibrium: repulsive brane. The hypersurface S corresponds to a Saddle point.
Constant warping factor:
Geodesic confinement in each leaf of the foliation.
Behaviour of the null geodesics when $f$ is a decreasing monotonic function

$$f(l) \to -\infty \text{ as } l \to \infty$$
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Classical and geometrical confinement mechanisms

How to mimic a confining quantum scalar field?

a) Through special interaction of the particle with a classical scalar field.

This is done, for instance, by defining the following action:
b) Modifying the geometry: for instance, assuming a Weyl geometry for the bulk.
In Weyl geometry we define a global 1-form \( W \), such that

\[
\frac{d}{d\lambda} g(\overline{V}, \overline{U}) = 0
\]

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\[
\frac{d}{d\lambda} g(\overline{V}, \overline{U}) = \widetilde{W} \left( \frac{d}{d\lambda} \right) g(\overline{V}, \overline{U})
\]
Weyl integrable geometry

Assume a global scalar field defined on the embedding manifold, such that

$$\widetilde{W} = d\phi$$

Two important facts:

It is possible to have a Riemannian brane embedded in the Weylian bulk!
The geodesic equations get modified:

In the case of a warped bulk the resulting effect is

\[ f \rightarrow f + \phi \]

This allows geodesical confinement!